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K Functionals and Best Polynomial Approximation in Weighted $L^p(R)$

Z. DITZIAN

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

AND

V. TOTIK

Bolyai Institute, Aradi V. tere 1, Szeged 6720, Hungary

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DEDICATED TO THE MEMORY OF GÉZA FREUD

Best polynomial approximation in weighted $L^p(R)$ spaces was a subject to which Géza Freud devoted much of the last 10 years of his life. This is described in H. N. Mhaskar's accurate account [3] and is illustrated by the numerous papers on the subject that Freud published, partly in association with Mhaskar, between 1972 and 1983 (the latest posthumously). Freud and Mhaskar developed a beautiful theory analogous to the classical theory of best trigonometric approximation. However, only analogues of the first and second moduli of smoothness were treated. This shortcoming is probably the reason why Mhaskar calls his description "the direct and converse theorem as they stand today." The aim of our paper is to fill this gap by providing a new modulus of smoothness which is not only defined for all orders but also (we believe) is more elegant for $r=1$ and $r=2$ than that of Freud an Mhaskar. We will show that our modulus of smoothness is equivalent to the K functional given in [2, (8)] and [3, (15)] and by that we will solve the characterization problem mentioned in Mhaskar's paper [3]. We emphasize that the really deep inequalities on the subject ((4) and (5) below) are due to Freud (see [1, 3]) and our contribution is limited to the description of the relevant K functional.

In our discussion of weighted $L^p(R)$ we shall restrict the weight to $W(x) = W_Q(x) = \exp(-Q(X))$, where $Q(x)$ satisfies the following conditions:

(a) Q is an even function and $Q(x) \in C^1(R_+)$,

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(b) $Q'(x) \nearrow \infty$ as $x \nearrow \infty$, and

(c) $Q'(x+1) \leqslant AQ'(x)$ for $x > 1$, for some constant A.

Examples of such $W(x)$ are $W(x) = \exp(-c|x|^{\beta})$ for $\beta > 1$ and $c > 0$. The weights in (a)–(c) are more general than those discussed by Freud and Mhaskar.

The K functional which we will characterize is given by:

$$K_r(f, t^r) = K_r(f, t^r)_{W,p} = \inf\{\| W(f-g)\|_p + t^r \| Wg^{(r)}\|_p;\ g^{(r-1)} \in \text{A.C.}_{\text{loc}}\} \tag{1}$$

where $g^{(r-1)} \in \text{A.C.}_{\text{loc}}$ means $g^{(r-1)}$ is absolutely continuous in every finite interval.

Let $t^* = t^*(t)$ be defined by $tQ'(t^*) = 1$ if this condition defines t^* uniquely; otherwise set $t^* = 0$. Clearly $t^*(t) \nearrow \infty$ as $t \searrow 0$. We now define our rth modulus of smoothness with the aid of t^* by

$$\begin{aligned}\omega_r(f, t) = \omega_r(f, t)_{W,p} = \sup_{0<h<t} &\| W \Delta_h^r f \|_{L^p[-h^*, h^*]} \\ &+ \| W(f - P_{r,t} f)\|_{L^p(t^*, \infty)} \\ &+ \| W(f - \tilde{P}_{r,t})\|_{L^p(-\infty, -t^*)}\end{aligned} \tag{2}$$

where $\Delta_h^r f(x) = \sum_{k=0}^{n} (-1)^k f(x + (r/2 - k)h)$ (the usual symmetric difference) and $P_{r,t} f$ $(\tilde{P}_{r,t} f)$ is the orthogonal projection of f in $L_W^2(t^*, \infty)$ $(L_W^2(-\infty, -t^*))$ onto π_{r-1}, the set of polynomials of degree at most $r-1$. We denote by $L_W^p(a, b)$ the set of all measurable functions satisfying $\| Wf \|_{L^p(a,b)} < \infty$. For $f \in L_W^p(R) \equiv L_W^p$ we have $\int_{-\infty}^{\infty} |f(x)|\, |x|^i (W(x))^2\, dx < \infty$ for all i, and therefore $P_{r,t} f$ and $\tilde{P}_{r,t} f$ are well defined.

THEOREM. *For every positive integer r and $1 \leqslant p \leqslant \infty$ there exists a constant C such that for every $f \in L_W^p$ and $0 < t \leqslant 1$ we have*

$$C^{-1}\omega_r(f, t) \leqslant K_r(f, t^r) \leqslant C\omega_r(f, t). \tag{3}$$

COROLLARY 1. *For* $\bar{\omega}_r(f, t) \equiv \sup_{0<h\leqslant t} \|\Delta_h^r f\|_{L^p[-h^*, h^*]} + \| f \|_{L^p(t^*, \infty)} + \| f \|_{L^p(-\infty, -t^*)}$ *there exist two positive constants c and C for which* $C^{-1}K_r(f, t^r) \leqslant \bar{\omega}_r(f, t) \leqslant CK_r(f, t^r) + Ce^{-c/t}\| Wf \|_p$ *for all* $f \in L_W^p$ *and* $0 < t \leqslant 1$.

Note that if f is not a polynomial of order at most $r-1$, then the term $e^{-c/t}\| Wf\|_p$ may be neglected, as in that case $K_r(f, t^r) \geqslant c_1 t^r$ for some $c_1 > 0$.

COROLLARY 2. *For $\tilde{\omega}_r(f, t) \equiv \sup_{0<h\leq t} \| W \Delta_h^r f \|_{L^p(-h^*, h^*)}$ there exist two positive constants c and C for which*

$$C^{-1}\tilde{\omega}_r(f, t) \leq K_r(f, t^r) \leq C \int_0^t \frac{\tilde{\omega}_r(f, \tau)}{\tau} d\tau + Ce^{-c/t} \| Wf \|_p$$

for every $f \in L_W^p$ and $0 < t \leq 1$. In particular, for $\alpha > 0$ $K_r(f, t^r) = \mathcal{O}(t^\alpha)$ and $\tilde{\omega}_r(f, t) = \mathcal{O}(t^\alpha)$ (or, which is the same, $\| W \Delta_h^r f \|_{L^p[-h^*, h^*]} = \mathcal{O}(h^\alpha)$) are equivalent.

Remarks. I. If Q satisfies (a) but instead of (b) and (c) we assume that Q' is bounded and continuous, then (3) holds with $\omega_r(f, t) = \operatorname{Sup}_{0<h\leq t} \| W \Delta_h^r f \|_p$.

II. The full force of the conditions on $Q(x)$ are needed only in Corollaries 1 and 2. The Theorem is valid if Q satisfies the conditions: (a)′ $Q(x)$ is an even function, $Q(x) \in C[0, \infty)$, (b)′ $Q(x+t) - Q(x) \nearrow \infty$ as $x \nearrow \infty$ for every $t > 0$, (c)′ $Q(x+1) = \mathcal{O}(Q(x))$. In this case t^* is given by

$$t^*(t) = \sup \{x; Q(x+rt) - Q(x) \leq 1\}.$$

III. $P_{r,t}$ (and $\tilde{P}_{r,t}$) are uniformly bounded projections of $L_W^p(t^*, \infty)$ ($L_W^p(-\infty, -t^*)$) onto π_{r-1} in L^p-norm. We could have exchanged $P_{r,t}$ for other (possibly sublinear) projections onto π_{r-1}, for example, for $P_{r,t}^*$ given by $\| W(f - P_{r,t}^*(f)) \|_{L^p(t^*, \infty)} = \inf_{P \in \pi_{r-1}} \| W(f-P) \|_{L^p(t^*, \infty)}$. The proof using $P_{r,t}^*$ would be simpler but $P_{r,t}$, being a Hilbert space projection on a fixed subspace, can be explicitly computed, and that advantage is rare in approximation theory.

We now turn to the implication of our theorem and its corollaries to weighted best polynomial approximation. Denote the best approximation of f by π_{n-1} in L_W^p by $E_n(f) = E_n(f)_{W,p} \equiv \inf_{P \in \pi_{n-1}} \| W(f-P) \|_p$. Freud proved with some additional assumptions on $Q(x)$ that

$$E_n(f) \leq C \frac{q_n}{n} E_{n-1}(f') \qquad \text{for} \quad f \in \text{A.C.}_{\text{loc.}} \tag{4}$$

and

$$\| WP' \|_p \leq C \frac{n}{q_n} \| WP \|_p \qquad \text{for} \quad P \in \pi_{n-1} \tag{5}$$

where $q_n Q'(q_n) = 1$ (and $q_n = 1$ when $q_n Q'(q_n) = 1$ does not define uniquely q_n). Using (4) and (5), standard K functional argument yields the estimates

$$E_n(f) \leq CK_r\left(f, \left(\frac{q_n}{n}\right)^r\right) \quad \text{and} \quad K_r\left(f, \left(\frac{q_n}{n}\right)^r\right) \leq C \left(\frac{q_n}{n}\right)^r \sum_{k=1}^n k^{r-1} q_k^{-r} E_k(f).$$

In view of our theorem, we can exchange here K_r for ω_r obtaining in this way the analogue of the classical Jackson–Stechkin estimates. As another consequence, we mention that for the weight $W(x) = \exp(-c|x|^\beta)$, $\beta \geq 2$, $c > 0$, Corollary 2 implies for $r > \alpha$ that $E_n(f) = \mathcal{O}(n^{(1-\beta)\alpha/\beta})$ is equivalent to

$$\| W \Delta_h^r f \|_{L^p[-h^{1/(1-\beta)}, h^{1/(1-\beta)}]} = \mathcal{O}(h^\alpha).$$

The proofs, together with other results concerning weighted polynomial approximation on finite intervals, will appear elsewhere.

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